JOURNAL OF ALGEBRA 62, 320–332 (1980)

Truncation and Duality in the Character Ring of a Finite Group of Lie Type*

CHARLES W. CURTIS

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

Communicated by I. N. Herstein

Received February 28, 1979

TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

INTRODUCTION

In this paper we consider two operations in the ring of complex valued characters of a finite group of Lie type G. The first, which we shall call truncation, is defined for each standard parabolic subgroup P_J, with unipotent radical V_J and Levi factor L_J. The operation assigns to each complex character ζ of G a character $\zeta_{(P_J/V_J)}$ of L_J, which is afforded by the submodule left fixed by V_J in a module affording ζ. In case the truncation $\zeta_{(P_J/V_J)}$ is known, we obtain formulas expressing the value of ζ at elements $x \in G$ such that $C_G(x) \leqslant L_J$, in terms of values of characters of L_J; more precisely, we have $\zeta(x) = \zeta_{(P_J/V_J)}(x)$ for such elements x.

The second operation is a generalization of the construction of the Steinberg character given in [2]. It assigns to each character ζ of G a virtual character ζ^*, called the dual of ζ, which is such that the dual 1_G^* of the principal character 1_G is the Steinberg character St_G. From Theorem B of [3], it follows that $(\mathrm{St}_G)_{(P_J/V_J)} = \mathrm{St}_{L_J}$. This can be interpreted as the statement that, in this case anyway, the operation of truncation intertwines the duality operation. The main result of this paper is a proof of this statement for all virtual characters of G. We also include results identifying ζ^* for irreducible characters ζ in $1_B{}^G$, and certain other principal series characters.

1. STATEMENT OF RESULTS

In this paper we are concerned with modules and characters of finite groups over the field of complex numbers C.

* This work was partially supported by National Science Foundation Grant MCS76-07015.

0021-8693/80/020320–13$02.00/0

We begin with some elementary considerations. Let X be a finite group, $Y \leqslant X$, and let H be a subgroup such that $Y \leqslant H \leqslant N_X(Y)$. Let W be a left CX-module, with character ω. The restriction $\omega|_H$ to H can be expressed in the form

$$\omega|_H = \omega'_H + \omega''_H ,$$

where ω'_H is the contribution to ω_H of the characters of H having Y in their kernel, and ω''_H is the contribution from those which do not.

On the other hand, let $\mathrm{inv}_Y(W)$ denote the subspace of W affording the trivial representation of Y. Then, because $Y \trianglelefteq H$, $\mathrm{inv}_Y(W)$ is a CH-submodule of $W|_H$, and we let $\omega_{(H)}$ denote the character of H afforded by $\mathrm{inv}_Y(W)$. As $Y \leqslant \ker \omega_{(H)}$, we can also view $\omega_{(H)}$ as a character $\omega_{(H/Y)}$ of the factor group H/Y.

We can now state the following result, whose proof is left as an exercise for the reader.

(1.1) PROPOSITION. *Let Y, X, H, W, ω be as above. Then:*

(i) $\omega'_H = \omega_{(H)} = (\omega|_H)_{(H)} = \sum (\omega, \tilde{\lambda}^X)\tilde{\lambda}$, *where the sum is taken over the set $\{\lambda\}$ of irreducible characters of H/Y, lifted to characters $\{\tilde{\lambda}\}$ of H with Y in their kernels*;

(ii) *let $h \in H$ be an element such that $C_X(h) \cap Y = \{1\}$. Then*

$$\omega(h) = \omega_{(H)}(h).$$

The second statement may be viewed as a kind of reduction formula, giving the values of the character ω on certain elements in terms of values of the truncated character $\omega_{(H)}$.

Let G be a finite group with a split (B, N)-pair of characteristic p, and Coxeter system (W, R), as in Chapter 1 of [3]. For each $J \subseteq R$, we let P_J denote the standard parabolic subgroup corresponding to J, and $P_J = L_J V_J$ the standard Levi decomposition of P_J, as in [3]. We shall apply the preceding considerations to characters of G, with $\{P_J, V_J\}$ playing the role of H and Y.

Let ζ be a character of G, afforded by a module Z. For each $J \subseteq R$, let $\zeta_{(P_J)}$ be the character of P_J afforded by $\mathrm{inv}_{V_J}(Z)$, and $\zeta_{(P_J/V_J)}$ the character of L_J defined by $\zeta_{(P_J)}$ (using the canonical isomorphism $L_J \cong P_J/V_J$). Extending the definition of $\zeta_{(P_J)}$ by additivity, we have a well-defined additive map $\zeta \to \zeta_{(P_J)}$ from $\mathrm{char}_Z(G) \to \mathrm{char}_Z(P_J)$, where $\mathrm{char}_Z(G)$ denotes the ring of virtual characters of G, etc. We shall call $\zeta_{(P_J/V_J)}$ a *truncation* of ζ.

A second operation in the character ring of G is defined as follows. Let $\zeta \in \mathrm{char}_Z(G)$, and define the *dual* ζ^* of ζ to be

$$\zeta^* = \sum_{J \subseteq R} (-1)^{|J|} \zeta^G_{(P_J)} \in \mathrm{char}_Z(G). \tag{1.2}$$

For example, from [2] we see that

$$1_G^* = \mathrm{St}_G ,$$

where St_G is the Steinberg character of G; conversely, $\mathrm{St}_G^* = 1_G$. On the other hand, if ζ is a cuspidal character, then $\zeta^* = \pm\zeta$.

For each $J \subseteq R$, the standard Levi factor L_J also has a split (B, N)-pair with Weyl group W_J. A set of standard parabolic subgroups of L_J is given by

$$Q_K = P_K \cap L_J , \qquad K \subseteq J$$

(see [3]). Thus for each virtual character μ of L_J, we can define the dual, again denoted by μ^*, where

$$\mu^* = \sum_{K \subseteq J} (-1)^{|K|} \mu_{(Q_K)}{}^{L_J} \in \mathrm{char}_Z(L_J).$$

The first main result is that the operations of forming duals and truncation intertwine each other. More precisely, we have

(1.3) THEOREM. *Let $\zeta \in \mathrm{char}_Z(G)$, and let $J \subseteq R$. Then*

$$(\zeta^*)_{(P_J/V_J)} = \zeta_{(P_J/V_J)}{}^* ,$$

i.e.

$$\sum_{J' \subseteq R} (-1)^{|J'|} (\zeta_{(P_{J'})}^G)_{(P_J/V_J)} = \sum_{K \subseteq J} (-1)^{|K|} (\zeta_{(P_J/V_J)})_{(Q_K)}^{L_J} .$$

In particular, this result implies that

$$(\mathrm{St}_G)_{(P_J/V_J)} = \mathrm{St}_{L_J} .$$

Using (ii) of Proposition 1.1, we have the following reduction theorem.

(1.4) COROLLARY. *Let $\zeta \in \mathrm{char}_Z(G)$, and let $J \subseteq R$. Let $x \in G$ be an element such that $C_G(x) \leqslant L_J$. Then*

$$\zeta^*(x) = \zeta_{(P_J/V_J)}{}^*(x).$$

Using Theorem 1.3, Alvis has proved that for an arbitrary finite group G of Lie type, as above, the duality operation $\zeta \to \zeta^*$ permutes up to sign the irreducible characters of G. In other words, if ζ is an irreducible character, then $\pm\zeta^*$ is an irreducible character. Examples where $-\zeta^*$ is an irreducible character are easy to find. For example in a group of rank two there exist irreducible characters ζ such that $\zeta_{(B)} = 0$ but $\zeta_{(P)} \neq 0$, where B is a Borel subgroup, and P a maximal parabolic subgroup, and for such a character, ζ^* is not a character.

We next state some results which show, among other things, that the map $\zeta \to \zeta^*$ does permute irreducible characters, at least, for certain characters in the principal series.

In order to state these results, we follow the terminology of Section 5 of [3]. We have a system $\mathscr{S}$ of groups with (B, N)-pairs $\{G(q)\}$. Each group $G(q)$ is assumed to have a split (B, N)-pair of characteristic $p \mid q$, and a fixed Coxeter system (W, R) (independent of q).

Let A be the generic ring associated with the system $\mathscr{S}$; then A is an algebra over the ring $\mathfrak{o}$, where $\mathfrak{o} = K[X]$. Let $F = Q(X)$, $\bar{F}$ a finite extension of F which splits A^F, and $\mathfrak{o}^*$ the integral closure of $\mathfrak{o}$ in $\bar{F}$. For each q, let $f^*\colon \mathfrak{o}^* \to \bar{Q}$ be an extension of the homomorphism $f\colon \mathfrak{o} \to Q$ given by $X \to q$, into the algebraic closure $\bar{Q}$ of Q.

Using the extensions $\{f^*\}$, there can be defined a bijection $\varphi \to \zeta_{\varphi,q}$ from the irreducible characters $\{\varphi\}$ of W to the irreducible components $\{\zeta_{\varphi,q}\}$ of $1_{B(q)}^{G(q)}$, and for each $J \subseteq R$, a bijection $\psi \to \eta_{\psi,q}$ from the irreducible characters of W_J to the irreducible components $\{\eta_{\psi,q}\}$ of $1_{B_J(q)}^{L_J(q)}$. By Theorem 5.1 of [3], these bijections satisfy, for all q, φ, and ψ as above,

$$(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)}) = (\varphi, \psi^W). \tag{1.5}$$

(For indecomposable Coxeter groups W, it was proved in [1] that the correspondence $\varphi \to \zeta_{\varphi,q}$ is canonical, i.e., independent of f^*, except for certain characters occurring in types ${}^2F_4$, E_7, and E_8.)

Following Section 5 of [3], there exists, for each irreducible character φ of W, a *generic degree* $d_\varphi(X) \in Q[X]$, such that for each q,

$$d_\varphi(q) = \zeta_{\varphi,q}(1).$$

Similarly we have generic degrees $d_\psi(X)$ corresponding to irreducible characters ψ of W_J, for $J \subseteq R$.

The next result is due to Green [5]. For completeness we shall give a new proof in Section 3. The homomorphisms IND and SGN from $A \to \mathfrak{o}$ are defined in [3].

(1.6) PROPOSITION (Green). (i) *Let $\{a_w\}_{w \in W}$ be the standard basis of A^F. There exists an involution of rings $J\colon A^F \to A^F$, given by*

$$J\left(\sum \lambda_w a_w\right) = \sum J_F(\lambda_w)\,\mathrm{SGN}(a_w) J_F(\mathrm{IND}(a_w)) a_w\,,$$

for $\{\lambda_w\}$ in F, where $J_F(\lambda_w)$ is the result of applying the automorphism $J_F\colon X \to X^{-1}$ of F to λ_w. The involution J is semilinear with respect to the automorphism J_F of F.

(ii) *The involution $J\colon A_F \to A_F$ permutes the irreducible characters of A^F in such a way that if χ corresponds to the character φ of W (as in* [3, Sect. 5]*), and $\chi \to {}^J\chi$, then ${}^J\chi$ corresponds to $\epsilon\varphi$, where ϵ is the sign character of W.*

(iii) *If $d_\varphi(X)$ is the generic degree corresponding to φ, then the generic degree corresponding to $\epsilon\varphi$ is $X^N d_\varphi(X^{-1})$, where $X^N = \mathrm{IND}(a_{w_0})$, and w_0 is the element of maximal length in W.*

We now have:

(1.7) THEOREM. *For all irreducible characters φ of W, and all q, we have*

$$\zeta^*_{\varphi,q} = \zeta_{\epsilon\varphi,q}\,,$$

where $\zeta^*_{\varphi,q}$ is the dual of $\zeta_{\varphi,q}$ defined in (1.2). *Thus the map $\zeta \to \zeta^*$ permutes the irreducible components of $1_{B(q)}^{G(q)}$.*

We conclude with a calculation of ζ^*, for certain irreducible characters not in $1_B{}^G$. Let G be a finite group with a split (B, N)-pair of characteristics p, as in the beginning of the discussion. Let $U = O_p(B)$, and assume that the root subgroups of U satisfy the hypothesis in [6, 3.9]. Let λ: $B \to C$ be a linear character of B with $U \leqslant \ker \lambda$. Then the induced character λ^G contains two noteworthy characters, the *generalized Steinberg character* $\mathrm{St}_{G,\lambda}$ (see [8]) and a character $\zeta(\lambda)$ defined by Howlett [6, 6.8]. As our final result we prove

(1.8) PROPOSITION. $\mathrm{St}^*_{G,\lambda} = \zeta(\lambda)$, *for all linear characters λ of B with $U \leqslant \ker \lambda$.*

Thus $\zeta(\lambda)$ plays the role of a generalized identity character.

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.3 will be given in a series of steps. By Proposition 1.1, a typical term in $(\zeta^*)_{(P_J/V_J)}$ can be expressed in the form

$$(\zeta^G_{(P_{J'})})_{(P_J/V_J)} = (\zeta^G_{(P_{J'})}\,|_{P_J})_{(P_J/V_J)}\,.$$

Changing the notation slightly in order to be able to refer conveniently to the results in Sections 1 and 2 of [3], we have, for J_1, $J_2 \subseteq R$,

$$\zeta^G_{(P_{J_2})}\,|_{P_{J_1}} = \sum_{x\in W_{J_1,J_2}} [{}^x\zeta_{(P_{J_2})}\,|_{{}^xP_{J_2}\cap P_{J_1}}]^{P_{J_1}}\,,$$

where W_{J_1,J_2} is the set of distinguished (W_{J_1}, W_{J_2}) double coset representatives in W. For a fixed $x \in W_{J_1,J_2}$, we have

$$W_{J_1} \cap {}^xW_{J_2} = W_K$$

for some $K \subseteq J_1$ (see [9], Lemma 2, or [7]). Then from [3, Sect. 2],

$$P_K = ({}^xP_{J_2} \cap P_{J_1})V_{J_1}, \qquad V_K = ({}^xV_{J_2} \cap P_{J_1})V_{J_1}.$$

Our first main objective is to prove

(2.1) PROPOSITION. *Let $J_1, J_2 \leqslant R$, $x \in W_{J_1,J_2}$, and K be as above. Then*

$$[{}^x\zeta_{(P_{J_2})} |_{{}^xP_{J_2} \cap P_{J_1}}]^{P_{J_1}}_{(P_{J_1}/V_{J_1})} = \zeta_{(P_{J_1}/V_{J_1})(Q_K)}{}^{L_{J_1}},$$

where $Q_K = P_K \cap L_{J_1}$ is the standard parabolic subgroup of L_{J_1} corresponding to K.

We begin the proof with the following result, which is perhaps of some independent interest, concerning the truncation operation with relation to pairs of associated parabolic subgroups.

(2.2) LEMMA. *Let P_{J_1} and P_{J_2} be standard parabolic subgroups such that $L_{J_1} = {}^wL_{J_2}$ for some $w \in W_{J_1,J_2}$. Then for each character ζ of G, $\zeta_{(P_{J_1}/V_{J_1})} = {}^w\zeta_{(P_{J_2}/V_{J_2})}$, where $\zeta_{(P_{J_1}/V_{J_1})}$ is viewed as a character of L_{J_1} and $\zeta_{(P_{J_2}/V_{J_2})}$ as a character of L_{J_2}.*

Prooof. By Proposition 1.1, we have

$$\zeta_{(P_{J_1}/V_{J_1})} = \sum_{\lambda} (\zeta, \tilde{\lambda}^G)\lambda,$$

and

$$\zeta_{(P_{J_2}/V_{J_2})} = \sum (\zeta, \tilde{\lambda}'^G)\lambda',$$

where the sum is taken over the irreducible characters of L_{J_1} and L_{J_2}, respectively. We can pair off the characters $\{\lambda\}$ of L_{J_1} and $\{\lambda'\}$ of L_{J_2} in such a way that ${}^w\lambda' = \lambda$. Thus it suffices to prove that if λ_1 and λ_2 are irreducible characters of L_{J_1} and L_{J_2}, with $\lambda_1 = {}^w\lambda_2$, then $\tilde{\lambda}_1{}^G = \tilde{\lambda}_2{}^G$. This was proved for cuspidal characters by Harish-Chandra (see [3, 3.5]), by showing that $(\tilde{\lambda}_1{}^G, \tilde{\lambda}_1{}^G) = (\tilde{\lambda}_1{}^G, \tilde{\lambda}_2{}^G) = (\tilde{\lambda}_2{}^G, \tilde{\lambda}_2{}^G)$. The proof given in [3, pp. 674–675] applies to the present situation with a few minor changes to fill in, which we leave as exercises for the reader.

(2.3) LEMMA. *Let X be a finite group, and let $V \trianglelefteq X$ and $H \leqslant X$. Let M be a left CH-module.*

(i) *There is an isomorphism of CX-modules*

$$\mathrm{inv}_V(M^X) \cong (\mathrm{inv}_V(M^{HV}))^X.$$

(ii) *There is an isomorphism of vector spaces*

$$(M^{HV})_{(HV/V)} \cong M_{(H/V \cap H)}$$

which intertwines the action of the groups HV/V and $H/V \cap H$, using the natural isomorphism between them.

Proof. For the first result, we express M^X as $(M^{HV})^X$, and the result follows immediately from the theory of induced modules (see [4, Sect. 12]). For the second statement, let $\{v_i\} \subset V$ be a transversal of HV/H, so that $M^{HV} = \bigoplus_i v_i \otimes_H M$. One then shows that $\mathrm{inv}_V(M^{HV})$ consists of the elements $(\sum v_i) \otimes m$, where $\{m\}$ ranges over $\mathrm{inv}_{V \cap H}(M)$. Moreover, if $h \in H$, then $hv_i = v_j h_j$ for some h_j, and it is easily checked that all h_j are $\equiv h \pmod{V \cap H}$. The obvious map from $\mathrm{inv}_V(M^{HV}) \to \mathrm{inv}_{V \cap H}(M)$ then intertwines the action of the corresponding elements of HV/V and $H/H \cap V$, as required.

Proof of (2.1). Let Z be a left CG-module affording ζ. The left side of (2.1) is afforded by the $C(P_J/V_J)$-module

$$\mathrm{inv}_{V_{J_1}}[(\mathrm{inv}_{{}^xV_{J_2}}(Z)|_{P_{J_1} \cap {}^xP_{J_2}})^{P_{J_1}}]. \tag{2.4}$$

Letting

$$V_{J_1} \to V, \quad P_{J_1} \cap {}^xP_{J_2} \to H, \quad \mathrm{inv}_{{}^xV_{J_2}}(Z)|_{P_{J_1} \cap {}^xP_{J_2}} \to M, \quad \text{and} \quad P_{J_1} \to X,$$

we deduce from Lemma 2.3 that the module in (2.4) is isomorphic to

$$\{[\mathrm{inv}_{V_{J_1}}[(\mathrm{inv}_{{}^xV_{J_2}}(Z)|_{P_{J_1} \cap {}^xP_{J_2}})^{P_K}]\}^{P_{J_1}} = Y^{P_{J_1}},$$

where $P_K = (P_{J_1} \cap {}^xP_{J_2})\, V_{J_1}$, and Y is the P_K-module in the brackets,

$$Y = \mathrm{inv}_{V_{J_1}}[\mathrm{inv}_{{}^xV_{J_2}}(Z)|_{P_{J_1} \cap {}^xP_{J_2}})^{P_K}].$$

We first note that every transversal for L_{J_1}/Q_K is a transversal for P_{J_1}/P_K. In fact, from [3, 2.5], we have $Q_K = P_K \cap L_{J_1} = {}^xP_{J_2} \cap L_{J_1}$, and $L_{J_1} \cap {}^xP_{J_2} = L_K(L_{J_1} \cap {}^xV_{J_2})$. Thus, if $L_{J_1} = \bigcup l_j(L_{J_1} \cap {}^xP_{J_2})$ (disjoint), then from (2.5) of [3], we obtain

$$P_{J_1} = L_{J_1}V_{J_1} = \bigcup l_j(L_K(L_{J_1} \cap {}^xV_{J_2}))V_{J_1} = \bigcup l_j P_K.$$

Finally, $l_jP_K = l_{j'}P_K$ implies that $l_j^{-1}l_{j'} \in L_{J_1} \cap P_K$, so the cosets $\{l_jP_K\}$ are distinct. Thus, for such a transversal $\{l_j\}$ of L_{J_1}/Q_K, we have

$$Y^{P_{J_1}} = \bigoplus_j l_j \otimes_{P_K} Y. \tag{2.5}$$

We now calculate Y. Using part (ii) of Lemma 2.3, the P_K/V_{J_1}-module Y can be identified with the $P_{J_1} \cap {}^xP_{J_2}/V_{J_1} \cap {}^xP_{J_2}$-module

$$\mathrm{inv}_{V_{J_1} \cap {}^xP_{J_2}}(\mathrm{inv}_{{}^xV_{J_2}}(Z)) = \mathrm{inv}_{(V_{J_1} \cap {}^xP_{J_2})\,{}^xV_{J_2}}(Z).$$

The module on the right-hand side is $\mathrm{inv}_{{}^xV_{K'}}(Z)$, where ${}^PK'$ is the standard parabolic subgroup $({}^{x^{-1}}P_{J_1} \cap P_{J_2})\, V_{J_2}$, and $V'_K = ({}^{x^{-1}}V_{J_1} \cap P_{J_2})\, V_{J_2}$. It affords the character ${}^x\zeta_{(P_{K'}/V_{K'})}$. Now ${}^xL_{K'} = L_K$ and ${}^x\zeta_{(P_{K'}/V_{K'})} = \zeta_{(P_K/V_K)}$ by Lemma 2.2. It follows that

$$Y \cong \mathrm{inv}_{V_K}(Z),$$

as a P_K-module. Since $V_{J_1} \leqslant V_K$, it follows that $\mathrm{inv}_{V_K}(Z) = \mathrm{inv}_{V_K}(\mathrm{inv}_{V_{J_1}}(Z))$, and hence $Y \cong (Z_{(P_{J_1}/V_{J_1})})_{(Q_K)}$ as a $C(P_K \cap L_{J_1})$-module. Substituting in (2.5) yields the result

$$[{}^xZ_{(P_{J_2})|{}^xP_{J_2} \cap P_{J_1}}]_{(P_{J_1}/V_{J_1})}^{P_{J_1}} \cong Z_{(P_{J_1}/V_{J_1})(Q_K)}^{L_{J_1}}$$

completing the proof.

Proof of Theorem 1.3. Letting $Q_K = P_K \cap L_J$ for $K \leqslant J$, we have, by Proposition 2.1,

$$\begin{aligned}(\zeta^*)_{(P_J/V_J)} &= \sum_{J' \subseteq R} (-1)^{|J'|} (\zeta^G_{(P_{J'})}\, |_{P_J})_{(P_J/V_J)} \\ &= \sum_{J' \subseteq R} (-1)^{|J'|} \sum_{x \in W_{J,J'}} [({}^x\zeta_{(P_{J'})}\, |_{{}^xP_{J'} \cap P_J})^{P_J}]_{(P_J/V_J)} \\ &= \sum_{J' \subseteq R} (-1)^{|J'|} \sum_{\substack{x \in W_{J,J'} \\ {}^xW_{J'} \cap W_J = W_K}} (\zeta_{(P_J/V_J)})_{(Q_K)}\,.\end{aligned}$$

Letting $a_{J'JK}$ be the number of elements $x \in W_{J,J'}$ such that ${}^xW_{J'} \cap W_J = W_K$, the above sum can be written in the form

$$(\zeta^*)_{(P_J/V_J)} = \sum_{K \subseteq J} \left(\sum_{J' \subseteq R} (-1)^{|J'|} a_{J'JK} \right) (\zeta_{(P_J/V_J)})_{(Q_K)}^{L_J}\,.$$

The proof of Theorem 1.3 will be completed as soon as we establish:

(2.5) LEMMA. *Let J', $J \subseteq R$, and let* $a_{J'JK} = \mathrm{card}\{x \in W_{J,J'} : {}^xW_{J'} \cap W_J = W_K\}$. *Then*

$$\sum_{J' \subseteq R} (-1)^{|J'|} a_{J'JK} = (-1)^{|K|}.$$

Proof. Following Solomon [9, p. 256], we consider elements $\{x_J : J \subseteq R\}$ in the rational group algebra of W. Then from [9, pp. 256, 257], the elements $\{x_J\}$ are linearly independent, and we have

$$x_{J'}x_J = \sum_{K \subseteq J} a_{J'JK} x_K\,.$$

Then

$$\sum_{J' \subseteq R} (-1)^{|J'|} x_{J'} x_J = \sum_K \left(\sum_{J'} (-1)^{|J'|} a_{J'JK} \right) x_K .$$

Using the formula for the element w_0 of maximal length,

$$w_0 = \sum_{J' \subseteq R} (-1)^{|J'|} x_{J'} ,$$

from [9, p. 263], the left side of the above expression becomes $w_0 x_J$. The result of the lemma will follow, by comparing coefficients of the linearly independent elements $\{x_K\}$, from the statement

$$w_0 x_J = \sum_{K \subseteq J} (-1)^{|K|} x_K . \tag{2.6}$$

From [9], we have $x_J = \sum_{w \in X_J} w$, where X_J is the distinguished transversal for W/W_J. The proof of (2.6) is a nice exercise on root systems, and goes as follows. Suppose $w \in X_J$; then $w(\alpha_i) > 0$ for all roots $\alpha_i \in \Pi_J$, where Π_J is the subset of the fundamental roots Π corresponding to J. The coefficient of $w_0 w$ in $w_0 x_J$ is 1. To find the coefficient on the right-hand side of (2.6), consider the subset of Π defined by

$$J^* = \{\alpha_j \in \Pi : w_0 w(\alpha_j) > 0\}.$$

Now, $\alpha_j \in \Pi_J^*$ if and only if $w(\alpha_j) < 0$, hence $J^* \cap J = \varnothing$, and $w_0 w \in X_K$ for $K \subseteq J$ if and only if $K = \varnothing$. Thus $w_0 w$ appears on the right-hand side of (2.6) with coefficient $(-1)^{|\varnothing|} = 1$.

If $w \notin X_J$, then $w(\alpha_i) < 0$ for some $\alpha_i \in \Pi_J$, and $w_0 w(\alpha_i) > 0$, so there is a nonempty subset $K \subseteq J$ for which $w_0 w \in X_K$. Then the coefficient of $w_0 w$ on the right-hand side is

$$\sum_{w_0 w \in X_K} (-1)^{|K|} = 0,$$

which agrees with the coefficient on the left side of (2.6), and completes the proof of the lemma.

3. Duality in $1_B{}^G$

We begin with a proof of Green's result (Proposition 1.6). The fact that. J: $A^F \to A^F$ is a semilinear automorphism of the algebra A^F is easily proved by noting that J is a semilinear automorphism of the vector space A^F, and preserves the multiplicative defining relations of the algebra A^F. To prove that

$J^2 = 1$, it is sufficient to check that $J^2(a_w) = a_w$ for all $w \in W$, and this follows because

$$J_F(\mathrm{IND}(a_w)) = \mathrm{IND}(a_w)^{-1} \qquad w \in W.$$

For the second statement, we first take a fixed extension of J_F to an involution $J_{\bar{F}}$ of the splitting field $\bar{F}$. Now let T: $A^{\bar{F}} \to M_n(\bar{F})$ be an irreducible $\bar{F}$-representation of $A^{\bar{F}}$, with character χ. For $a \in A^F$, define

$$T^J(a) = J_{\bar{F}}(T(J(a))),$$

where $J_{\bar{G}}(T(J(a)))$ is the matrix obtained by applying $J_{\bar{F}}$ to all the entries of the matrix $T(J(a))$. Then T^J defines an $\bar{F}$-representation of $A^{\bar{F}}$ depending only on the equivalence class of T and the extension $J_{\bar{F}}$. Moreover, T^J is irreducible, and its character χ^J satisfies

$$\chi^J(a) = J_{\bar{F}}(\chi(J(a))), \qquad a \in A^F.$$

Thus $\chi \to \chi^J$ is a permutation of the irreducible characters of $A^{\bar{F}}$. (In case χ is a rational character of $A^{\bar{F}}$ in the sense that $\chi(a_w) \in \mathfrak{o}$ for all $w \in W$, the definition of χ^J is independent of the choice of an extension of J to $\bar{F}$. See [1] for further discussion of this point.)

We now have to show that if f^*: $\mathfrak{o}^* \to \bar{Q}$ is an extension of f: $X \to 1$, and $f^*(\chi(a_w)) = \varphi(w)$, $w \in W$, for a given irreducible character φ of W, then $f^*(\chi^J(a_w) = \epsilon\varphi(w)$ for all $w \in W$. We have

$$\chi^J(a_w) = \mathrm{SGN}(a_w)\, J_F{}^2(\mathrm{IND}(a_w))\, \chi(a_w), \qquad w \in W,$$

hence

$$f^*(\chi^J(a_w)) = \mathrm{SGN}(a_w) f^*(\chi(a_w) = \epsilon(w)\, \varphi(w),$$

as required.

Finally, the generic degree $d_\varphi(X)$, for an irreducible character φ of W corresponding to a character χ of $A^{\bar{F}}$, is given by

$$d_\varphi(X) = P(X) \deg \chi \left\{\sum \mathrm{IND}(a_w)^{-1} \chi(a_w)\, \chi(a_{w^{-1}})\right\}^{-1},$$

where $P(X) = \sum_w \mathrm{IND}(a_w)$ (see [1, (2.4)]). Clearly $X^N J_F(P(X)) = X^N P(X^{-1}) = P(X)$. Moreover, from the result of part (ii) of the proposition, we have

$$d_{\epsilon\varphi}(X) = P(X) \deg \chi^J \left\{\sum \mathrm{IND}(a_w)^{-1} \chi^J(a_w)\, \chi^J(a_{w^{-1}})\right\}^{-1}.$$

Using these facts, a simple calculation shows that $d_{\epsilon\varphi}(X) = X^N d_\varphi(X^{-1})$, and the proposition is proved.

We now turn to the proof of Theorem 1.7. We shall first prove that if $\zeta_{\varphi,q}$ and $\zeta_{\varphi',q}$ are irreducible characters in $1_{B(q)}^{G(q)}$, then for all $J \subseteq R$, we have

$$((\zeta_{\varphi,q})_{(P_J(q))}^{G(q)}, \zeta_{\varphi',q}) = (\varphi\,|_{W_J}^{W}, \varphi')_W . \tag{3.1}$$

By Proposition 1.1(i), and Lemma 5.1 of [3]

$$(\zeta_{\varphi,q})_{(P_J(q))} = \sum_{\psi} (\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)})\, \tilde{\eta}_{\psi,q}, \tag{3.2}$$

where the sum is taken over the irreducible characters ψ of W_J. From (1.5) above, we have

$$(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)}) = (\varphi, \psi^W).$$

Combining these results, we obtain, for the left-hand side of (3.1),

$$\sum_{\psi} (\varphi, \psi^W)(\psi^W, \varphi') = \left(\left[\sum_{\psi} (\varphi, \psi^W)\psi\right]^W, \varphi'\right).$$

By Frobenius reciprocity, we have

$$\sum_{\psi} (\varphi, \psi^W)\psi = \sum_{\psi} (\varphi\,|_{W_J}, \psi)\psi = \varphi\,|_{W_J},$$

and upon substituting this result in the previous formula, we obtain (3.1).

Using (3.1), we obtain

$$\begin{aligned}(\zeta_{\varphi,q}^{*}, \zeta_{\epsilon\varphi,q}) &= \sum_{J \subseteq R} (-1)^{|J|}((\zeta_{\varphi,q})_{(P_J(q))}^{G(q)}, \zeta_{\epsilon\varphi,q}) \\ &= \sum_{J} (-1)^{|J|}(\varphi\,|_{W_J}^{W}, \epsilon\varphi).\end{aligned}$$

Now $\epsilon\varphi$ is an irreducible character of W, and from Solomon's formula, $\epsilon = \sum (-1)^{|J|}\, 1_{W_J}^{W}$. Hence

$$\epsilon\varphi = \sum_{J} (-1)^{|J|}\varphi \cdot 1_{W_J}^{W} = \sum_{J} (-1)^{|J|}\varphi\,|_{W_J}^{W}.$$

Thus $(\zeta_{\varphi,q}^{*}, \zeta_{\epsilon\varphi,q}) = 1$. The same calculation shows that $(\zeta_{\varphi,q}^{*}, \zeta) = 0$ for all irreducible characters in $1_{B(q)}^{G(q)}$ different from $\zeta_{\varphi\epsilon,q}$. The proof of Theorem 1.7 will be completed if we can show that $((\zeta_{\varphi,q})_{(P_J(q))}, \zeta\,|_{P_J(q)}) = 0$ for all irreducible characters ζ of $G(q)$ which are not in $1_{B(q)}^{G(q)}$. From (3.2), it is sufficient to note that if, for some irreducible character ψ of W_J, $(\eta_{\psi,q}, \zeta\,|_{P_J(q)}) \neq 0$, then $\zeta \in 1_{B(q)}^{G(q)}$. This completes the proof of Theorem 1.7.

4. The Generalized Steinberg Character and Its Dual

We first recall the construction of the character $\zeta(\lambda)$ of Howlett [6]. Following [6], we begin with a fixed set of nontrivial linear characters $\{\mu_i\}$ on the root subgroups $\{U_{-\alpha_i}\}$, $1 \leqslant i \leqslant n$, where $\{\alpha_1, ..., \alpha_n\}$ is the set of fundamental roots Π. For each $J \subseteq R$, let w_J be the element of maximal length in the Coxeter group W_J. Let μ_J: ${}^{w_J}U \to C$ be the linear character such that

$$\begin{aligned} \mu_J &= \mu_i \quad \text{on} \quad U_{-\alpha_i}, \quad \alpha_i \in \Pi_J, \\ \mu_J &= 1 \quad \text{on} \quad U_{w_J(\alpha)}, \quad \alpha > 0, \quad \alpha \notin \Pi_J. \end{aligned}$$

Howlett proved [6, Lemma 4.1] that for every $J \subsetneq R$, there is a unique irreducible character χ_J appearing with positive multiplicity in both $\mu_J^{P_J}$ and λ^{P_J}, where λ is a fixed linear character of B with U in the kernel of λ. In particular, there is a unique irreducible character $\mathrm{St}_{G,\lambda}$ of G appearing in both λ^G and the *Gelfand–Graev character* $\mu_R{}^G$. (Note that in general, the Gelfand–Graev character depends on the choice of the linear characters $\{\mu_i\}$; see [10] for conditions for its independence of the choice of the $\{\mu_i\}$.)

Now let T_J be a transversal for $S\backslash W/W_J$, where $S = \mathrm{Stab}_W(\lambda)$. For $J \subseteq R$ and $t \in T_J$, let $\chi_{J,t}$ be the unique irreducible character of P_J in $(\lambda^t)^{P_J}$ and $\mu_J^{P_J}$, and put $\varphi_J = \sum_{t\in T_J} \chi_{J,t}^G$. Then define

$$\zeta(\lambda) = \sum (-1)^{|J|}\varphi_J.$$

Howlett proved [6, (6.8), (6.9)] that $\pm\zeta(\lambda)$ is an irreducible character of G. We shall prove Proposition 1.8, which asserts that $\zeta(\lambda)$ is the dual $\mathrm{St}_{G,\lambda^*}$ of the generalized Steinberg character. From the definition of $\zeta(\lambda)$, it is sufficient to prove:

(4.1) Lemma. *For all $J \subseteq R$, we have*

$$(\mathrm{St}_{G,\lambda})_{(P_J)} = \sum_{t\in T_J} \chi_{J,t}.$$

Proof. From (1.1) we have

$$(\mathrm{St}_{G,\lambda})_{(P_J)} = \sum_{0} (\xi^G, \mathrm{St}_{G,\lambda})\xi,$$

where ξ ranges over the irreducible characters of P_J such that $V_J \leqslant \ker \xi$. Thus it is sufficient to prove that $(\chi_{J,t}^G, \mathrm{St}_{G,\lambda}) = 1$ for all $t \in T_J$, and that for characters ξ of P_J with $V_J \leqslant \ker \xi$, $(\xi^G, \mathrm{St}_{G,\lambda}) \neq 0$ implies $\xi = \chi_{J,t}$ for some $t \in T_J$.

We first note that for all $t \in T_J$,

$$\chi_{J,t} \in (\lambda^t)^{P_J}.$$

Hence $\chi_{J,t} \in 1_U^{P_J}$, and in particular, $\chi_{J,t} \in 1_{V_J}^{P_J}$, so that $V_J \subseteq \ker \chi_{J,t}$. Next, from Lemma 6.7 of [6], we have

$$(\chi_{J,t}^G, \mathrm{St}_{G,\lambda}) = (1_{W_J^t \cap S}^S, 1_S) = 1.$$

Now let ξ be an irreducible character of P_J with $V_J \leqslant \ker \xi$, such that $(\xi^G, \mathrm{St}_{G,\lambda}) \neq 0$. Then $(\xi^G, \lambda^G) \neq 0$, and by Theorem 4.4 of [3], $\xi_{(P_J/V_J)}$ is a principal series character of L_J. It follows that $\xi \in \theta^{P_J}$, for some linear character θ of B with $U \leqslant \ker \theta$. Then $(\theta^G, \lambda^G) \neq 0$, and by Theorem 3.5 of [3], $\theta = \lambda^w$ for some element $w \in W$. Then $(\xi, (\lambda^w)^{P_J}) \neq 0$, and hence $\xi \in (\lambda^t)^{P_J}$ for some element $t \in T_J$.

We next observe that $(\xi^G, \mathrm{St}_{G,\lambda}) \neq 0$ also implies that $(\xi^G, {\mu_R}^G) \neq 0$. By Rodier's theorem [10, Sect. 6, Proposition 6.5] it follows that $(\xi, \mu_J^{P_J}) \neq 0$. Thus ξ appears with positive multiplicity in both $(\lambda^t)^{P_J}$ and $\mu_J^{P_J}$, for some $t \in T_J$, and by Lemma 4.1 of [6], we have $\xi = \chi_{J,t}$, completing the proof.

References

1. C. T. Benson and C. W. Curtis, On the degrees and rationality of certain characters of finite Chevalley groups, *Trans. Amer. Math. Soc.* **165** (1972), 251–273; **202** (1975), 405–406.
2. C. W. Curtis, The Steinberg character of a finite group with a (B, N)-pair, *J. Algebra* **4** (1966), 433–441.
3. C. W. Curtis, Reduction theorems for characters of finite groups of Lie type, *J. Math. Soc. Japan* **27** (1975), 666–688.
4. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative algebras," Wiley–Interscience, New York, 1962.
5. J. A. Green, On the Steinberg characters of finite Chevalley groups, *Math. Z.* **117** (1970), 272–288.
6. R. B. Howlett, Some irreducible characters of Groups with $B\check{N}$ pairs, *J. Algebra* **39** (1976), 571–592.
7. R. Kilmoyer, "Some Irreducible Complex Representations of a Finite Group with a (B, N)-Pair," Ph.D. dissertation, MIT, 1969.
8. R. Kilmoyer, Principal series representations of finite Chevalley groups, *J. Algebra*, in press.
9. L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra* **41** (1976), 255–268.
10. T. A. Springer, Caracteres de groupes de Chevalley finis, *Sem. Bourbaki* (1972/73), No. 429.